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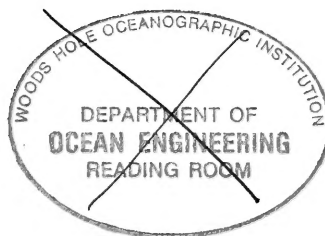
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Analytical Approximation for Steady Ship Waves at Low Froude Numbers

by
Francis Noblesse

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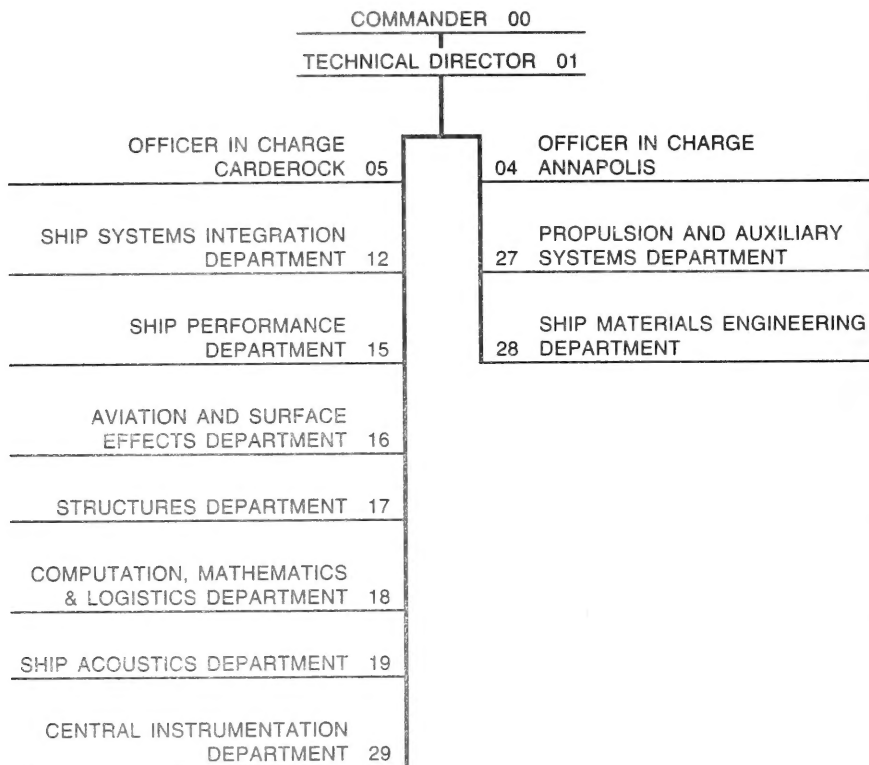
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   A simple analytical relationship between a ship-hull form and its steady far-field Kelvin wake is obtained by considering the low-Froude-number limit of the Neumann-Kelvin theory. In particular, this relationship predicts the occurrence of a sharp peak in the amplitude of the waves in the far-field Kelvin wake at an angle, α, from the ship track that is smaller than the Kelvin-cusp angle of $19°1/2$ for a hull form which has a small region of flare and is wall sided elsewhere, if the Froude number is sufficiently small. An explicit relationship between the angle, ϕ, between the ship track and the ship mean waterline in the region of flare and the corresponding "wave-peak" angle α in the Kelvin wake is obtained. For instance, this relationship predicts the occurrence of a sharp peak in wave amplitude at an angle α in the Kelvin wake equal to $14°$ for a hull having a small region of flare within which the waterline-tangent angle ϕ is approximately

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equal to either 30º or 74º. This theoretical result may explain the bright returns that have sometimes been observed in SAR images of ship wakes at angles smaller than the Kelvin-cusp angle. The low-Froude-number asymptotic analysis of the Neumann-Kelvin theory presented in this study also predicts that the wave-resistance coefficient is $O(F^2)$, where F is the Froude number, for a ship form with a region of flare, $O(F^4)$ for a ship form that is wall sided everywhere and has either a bow or a stern (or both) that is neither cusped nor round, and $O(F^6)$ for a wall-sided ship form with both bow and stern that are either cusped or round.

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ANALYTICAL APPROXIMATION FOR STEADY SHIP WAVES
AT LOW FROUDE NUMBERS

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ABSTRACT

A simple analytical relationship between a ship-hull form and its steady far-field Kelvin wake is obtained by considering the low-Froude-number limit of the Neumann-Kelvin theory. In particular, this relationship predicts the occurrence of a sharp peak in the amplitude of the waves in the far-field Kelvin wake at an angle, α, from the ship track that is smaller than the Kelvin-cusp angle of 19°1/2 for a hull form which has a small region of flare and is wall sided elsewhere, if the Froude number is sufficiently small. An explicit relationship between the angle, φ, between the ship track and the tangent to the ship mean waterline in the region of flare and the corresponding "wave-peak" angle α in the Kelvin wake is obtained. For instance, this relationship predicts the occurrence of a sharp peak in wave amplitude at an angle α in the Kelvin wake equal to 14° for a hull having a small region of flare within which the waterline-tangent angle φ is approximately equal to either 30° or 74°. This theoretical result may explain the bright returns that have sometimes been observed in SAR images of ship wakes at angles smaller than the Kelvin-cusp angle. The low-Froude-number asymptotic analysis of the Neumann-Kelvin theory presented in this study also predicts that the wave-resistance coefficient is $O(F^2)$, where F is the Froude number, for a ship form with a region of flare, $O(F^4)$ for a ship form that is wall sided everywhere and has either a bow or a stern (or both) that is neither cusped nor round, and $O(F^6)$ for a wall-sided ship form with both bow and stern that are either cusped or round.

1. INTRODUCTION

It has been observed, see for instance Fu and Holt (1982) and McDonough et. al. (1985), that SAR (Synthetic Aperture Radar) images of ship wakes sometimes reveal bright returns along rays at angles from the track of the ship smaller than the Kelvin-cusp angle of 19°1/2. A plausible explanation for these surprising observations was proposed by Scragg (1983) who considered a simple ship bow form with large flare for which he found that the zeroth-order slender-ship approximation to the far-field wave-amplitude function given in Noblesse (1983) predicted a sharp peak in the value of the amplitude of the divergent waves at an angle from the track of the ship equal to approximately half the bow entrance angle. This numerical result of Scragg was confirmed by Barnell and Noblesse (1986) who also found that the peak in the amplitude of the divergent waves becomes sharper as the value of the Froude number decreases, and thus suggested that the occurrence of a sharp peak in the amplitude of the far-field Kelvin waves was a large-flare low-Froude-number feature.

The numerical studies of Scragg and of Barnell and Noblesse are based on two simple approximations to the far-field wave-amplitude function, namely the Michell thin-ship approximation for which no peak was found and the zeroth-order slender-ship approximation which exhibited a peak as was already noted, so that it is not clear from these studies whether a more realistic mathematical model for the far-field wave-amplitude function, such as that provided by the Neumann-Kelvin theory, would also predict the occurrence of peaks in the

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amplitude of the far-field Kelvin waves. Furthermore, the numerical results obtained by Scragg and by Barnell and Noblesse correspond to a particular ship form and thus provide little physical insight into the origin of the predicted peak in the amplitude of the far-field Kelvin waves, specifically the manner in which such a peak is related to the shape of the ship hull.

A complementary analytical study of the low-Froude-number limit of the Neumann-Kelvin theory for an arbitrary ship form is thus presented here. This asymptotic analysis of the Neumann-Kelvin theory provides a simple analytical relationship between a ship-hull form and its steady far-field Kelvin wake. In particular, this relationship predicts the occurrence of a sharp peak in the amplitude of the waves in the far-field Kelvin wake at an angle, α, from the ship track that is smaller than the Kelvin-cusp angle of 19°1/2 for a hull form which has a small region of flare and is wall sided elsewhere, if the value of the Froude number is sufficiently small. A simple explicit relationship between the angle, φ, between the ship track and the tangent to the ship mean waterline in the region of flare and the corresponding wave-peak angle α in the Kelvin wake is given and depicted in Figure 3b. For instance, this figure predicts the occurrence of a sharp peak in the amplitude of the divergent or transverse waves at an angle α in the Kelvin wake equal to 14° for a hull having a small region of flare within which the waterline-tangent angle φ is approximately equal to 30° or 74°, respectively. This analytical result may explain the bright returns that have sometimes been observed in SAR images of ship wakes at angles smaller than the Kelvin-cusp angle.

The low-Froude-number asymptotic analysis of the Neumann-Kelvin theory presented in this study also predicts that the wave-resistance coefficient is $O(F^2)$, where F is the Froude number, for a ship form with a region of flare, $O(F^4)$ for a ship form that is wall sided everywhere and has either a bow or a stern (or both) that are neither cusped nor round, and $O(F^6)$ for a wall-sided ship form with both bow and stern that are either cusped or round.

More precisely, the low-Froude-number asymptotic approximation (50) to the far-field wave-amplitude function K(t) shows that the main contributions to the function K(t) stem from several particular points on the mean waterline. These are the bow and the stern, on one hand, and (usually but not always) one (or several) point(s) of stationary phase. Indeed, the number of these points of stationary phase, and their position on the waterline, depend on the value of t and on the shape of the waterline. The first two terms in the low-Froude-number asymptotic expansions for the contributions $K_{B,S}$ of the bow and stern and the contributions $K_{\pm}$ of the points of stationary phase in equation (50) are given by equations (51)-(55) and (59)-(62), respectively. The second-order terms in these asymptotic expansions are defined by complex expressions. However, the first-order terms provide simple approximations defined explicitly in terms of the geometrical characteristics of the hull and the velocity components in the tangential directions $\overline{t}$ and $\overline{n} \times \overline{t}$ to the hull (see Figure 2). In particular, these low-Froude-number asymptotic expansions show that the contributions K_B and K_S of the bow and stern are O(1) except if the bow or stern is cusped or round, in which case we have $K_{B,S} = O(F^2)$. The contribution of a given point of stationary phase is $O(1/F)$, and thus is dominant, if the hull has flare at this point; otherwise, that is if the hull is wall sided at the point of stationary phase, its contribution is $O(F)$. The low-Froude-number approximation (50) also shows that we have K(t) = $O(1/t^3)$ as t → ∞. In fact, this result is valid for any value of the Froude number.

The latter result implies that the lines along which the steepness of the short divergent waves in the far-field Kelvin wake takes given large values, say 1/7 and 1/15, are parallel to the ship track, as was found in Figure 21 of Barnell and Noblesse (1986) by using the Michell thin-ship approximation for a simple ship form. The Neumann-Kelvin theory therefore predicts that the far-field Kelvin wake contains three distinct regions: (i) a narrow constant-width inner region bordering the track of the ship where no divergent gravity waves can exist, (ii) an outer region where the usual transverse and divergent waves are present, and (iii) an intermediate region at the boundary between the inner and outer regions where short steep divergent waves can be found. In reality, surface tension and possibly also viscosity must evidently be taken into account in the vicinity of the track of the ship.

2. STATIONARY-PHASE APPROXIMATION TO THE FAR-FIELD KELVIN WAKE

The far-field Kelvin wake may be conveniently analyzed in terms of the nondimensional far-field coordinates $(\xi,\eta,\zeta) = (X,Y,Z)\,g/U^2$, where g is the gravitational acceleration, U is the speed of advance of the ship, and (X,Y,Z) are dimensional coordinates. The mean free surface is taken as the plane $\zeta = 0$, with the ζ axis pointing upwards, and the ξ axis is chosen along the track of the ship, that is in the ship centerplane, and pointing towards the bow. The origin of the system of coordinates is placed within the ship.

Equations (4), (3a), (7) and (8) in Barnell and Noblesse (1986) yield the following expression for the elevation ζ of the free surface at a sufficiently large distance behind the ship, such that nonlinearities may be neglected:

$$\pi\zeta(\xi,\eta) = \text{Re} \int_0^\infty (E_+ + E_-)\, K(t)\, (1+t^2)^{1/2}\, dt, \tag{1}$$

where K(t) is the far-field wave-amplitude function and $E_\pm$ is the exponential function defined as

$$E_\pm(t;\xi,\sigma) = \exp[i\xi\theta_\pm(t;\sigma)], \tag{2}$$

with the phase-function $\theta_\pm$ and the parameter σ defined as

$$\theta_\pm(t;\sigma) = (1\mp\sigma t)(1+t^2)^{1/2}, \tag{3}$$

$$\sigma = \eta/(-\xi). \tag{4}$$

For a ship with port- and starboard-symmetry, as is considered here, the Kelvin wake is symmetric about the ship track $\eta = 0$. We may then restrict the analysis of the Kelvin wake to the quadrant $\eta \geq 0$, $\xi < 0$ and assume $\sigma \geq 0$. Let α be the angle between the track of the ship and the line joining the origin of the system of coordinates to the observation point (ξ,η). We thus have

$$\alpha = \tan^{-1}\sigma. \tag{5}$$

A far-field asymptotic approximation, valid as $\xi \to -\infty$, to the wave integral (1) may be obtained by applying the method of stationary phase, as is well known. The result of this classical asymptotic analysis may be found in Barnell and Noblesse (1986), for instance. Specifically, equations (28), (24a), (20b), (25a-d) and (26a-d) in this reference yield

$$(-\xi)^{1/2}\zeta(\xi,\alpha) \sim \text{Re}$$

$$\{A_- K(t_-) \exp[i(\xi\Theta_- - \pi/4)]$$

$$+ \ A_+ K(t_+) \exp[i(\xi\Theta_+ + \pi/4)]\} \text{ as } \xi \to -\infty, \tag{6}$$

with $0 \leq \alpha < \tan^{-1}(2^{-3/2}) \simeq 19°28'$,

where $t_\pm(\alpha)$ are the values of t for which the phase-function $\theta_+(t;\alpha)$ is stationary, $\Theta_\pm(\alpha) = \theta_+(t_\pm;\alpha)$ represents the corresponding values of the phase-function $\theta_+(t;\alpha)$, and $A_\pm(\alpha)$ is the function defined as $A_\pm(\alpha) = (2/\pi)^{1/2}(1+t_\pm^2)^{1/2}/[\mp\theta_+''(t_\pm;\alpha)]^{1/2}$. The functions $t_\pm(\alpha)$, $\Theta_\pm(\alpha)$ and $A_\pm(\alpha)$ are given by

$$t_\pm(\alpha) = [1\pm(1-8\sigma^2)^{1/2}]/4\sigma, \tag{7}$$

$$\Theta_\pm(\alpha) = [3\mp(1-8\sigma^2)^{1/2}]$$
$$[1+4\sigma^2\pm(1-8\sigma^2)^{1/2}]^{1/2}/2^{1/2}8\sigma, \tag{8}$$

$$A_\pm(\alpha) = [1+4\sigma^2\pm(1-8\sigma^2)^{1/2}]^{3/4}2^{1/4}/$$
$$4\pi^{1/2}\sigma^{3/2}(1-8\sigma^2)^{1/4}, \tag{9}$$

where we have

$$\sigma = \tan\alpha \tag{10}$$

as is given by equation (5).

The far-field asymptotic approximation (6) shows that the wave pattern at any point (ξ,α), with $\xi \ll -1$ and $0 \leq \alpha < 19°28'$, consists in two elementary plane progressive waves, so-called transverse and divergent waves, as is most well known. The wavelengths $\lambda_\pm$ and the directions of propagation $\beta_\pm$, measured from the track of the ship, of the transverse (λ_-,β_-) and divergent (λ_+,β_+) waves at an angle α from the track of the ship are given by

$$\lambda_\pm(\alpha) = 2^{1/2}16\pi\sigma^2/[3\mp(1-8\sigma^2)^{1/2}]$$
$$[1-4\sigma^2\pm(1-8\sigma^2)^{1/2}]^{1/2}, \tag{11}$$

$$\beta_\pm(\alpha) = \cos^{-1}\{2^{1/2}2\sigma/[1+4\sigma^2\pm(1-8\sigma^2)^{1/2}]^{1/2}\}. \tag{12}$$

We have

$$1 \geq \lambda_-/2\pi \geq 2/3 \geq \lambda_+/2\pi \geq 0 \text{ and} \tag{13}$$

$$0 \leq \beta_- \leq \sin^{-1}(1/3^{1/2}) \simeq 35°16' \leq \beta_+ \leq 90°. \tag{14}$$

Equation (6) shows that the amplitudes of the transverse and divergent waves in the Kelvin wake are asymptotically equal to $A_\pm|K(t_\pm)|/(-\xi)^{1/2}$ as $\xi \to -\infty$. The steepnesses, say $s_\pm$, of these waves then are given by $s_\pm = A_\pm|K(t_\pm)|/\lambda_\pm(-\xi)^{1/2}$. We then have

$$(-\xi)^{1/2}s_\pm(\xi,\alpha) \sim S_\pm|K(t_\pm)| \text{ as } \xi \to -\infty, \tag{15}$$

where the functions $S_\pm(\alpha)$ are defined as $S_\pm(\alpha) = A_\pm(\alpha)/\lambda_\pm(\alpha)$. Equations (9) and (11) then yield

$$S_\pm(\alpha) = [3\mp(1-8\sigma^2)^{1/2}]\,[1-4\sigma^2\pm(1-8\sigma^2)^{1/2}]^{1/2}$$
$$[1+4\sigma^2\pm(1-8\sigma^2)^{1/2}]^{3/4}/2^{1/4}64\pi^{3/2}\sigma^{7/2}(1-8\sigma^2)^{1/4}. \tag{16}$$

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Equations (7), (8), (9), (10) and (16) show that we have

$$t_- = 0, \quad \Theta_- = 1, \quad A_- = (2/\pi)^{1/2} \qquad (17a,b,c)$$

$$\text{and } S_- = 1/2^{1/2}\pi^{3/2} \text{ for } \alpha = 0, \qquad (17d)$$

$$t_+ \sim 1/2\alpha, \quad \Theta_+ \sim 1/4\alpha, \quad A_+ \sim 1/2\pi^{1/2}\alpha^{3/2} \qquad (18a,b,c)$$

$$\text{and } S_+ \sim 1/16\pi^{3/2}\alpha^{7/2} \text{ as } \alpha \to 0, \qquad (18d)$$

$$t_\pm = 1/2^{1/2} \text{ and } \Theta_\pm = 3(3/2)^{1/2}/4 \qquad (19a,b)$$

$$\text{for } \alpha = \tan^{-1}(1/2^{3/2}) \simeq 19°28',$$

$$A_\pm \sim 3/\pi^{1/2}6^{1/4}(1-8\sigma^2)^{1/4} \text{ and } \qquad (19c)$$

$$S_\pm \sim 9/4\pi^{3/2}6^{1/4}(1-8\sigma^2)^{1/4} \qquad (19d)$$

$$\text{as } \alpha \to \tan^{-1}(1/2^{3/2}).$$

The stationary-phase values $t_\pm(\alpha)$, the phase-functions $\Theta_\pm(\alpha)$, the amplitude-functions $A_\pm(\alpha)$ and the steepness-functions $S_\pm(\alpha)$ are depicted in Figures 1a and 1b, which correspond to the transverse and divergent waves, respectively.

It may be seen from these figures and from equations (19c) and (19d) that the amplitude-functions $A_\pm(\alpha)$ and the steepness-functions $S_\pm(\alpha)$ are singular at the Kelvin cusp line $\sigma^2 = 1/8$, $\alpha = \tan^{-1}(1/2^{3/2}) \simeq 19°28'$, in accordance with the well-known fact that the asymptotic approximation (6) is not uniformly valid at the boundary of the Kelvin wake. A complementary asymptotic approximation, expressed in terms of Airy functions, valid at and near the Kelvin cusp line is given in Ursell (1960) for the particular case of the Kelvin wave pattern due to a concentrated pressure point at the free surface. However, Ursell's more complex asymptotic approximation will not be considered here because the simple asymptotic approximation (6) is little affected by the weak singularity $(1-8\sigma^2)^{-1/4}$ for points (ξ,α) inside the Kelvin wake and not too near the cusp line $\alpha \simeq 19°28'$, in which we are mostly interested in this study.

Equation (11) yields

$$\lambda_+ \sim 8\pi\alpha^2 \text{ as } \alpha \to 0. \qquad (20)$$

This approximation and the approximation (18b) show that there are an infinite number of divergent waves with indefinitely shorter wavelength in the vicinity of the track of the ship, as is well known. It may be seen from Figure 1b and equations (18a,c,d) that the stationary-phase value $t_+(\alpha)$, the amplitude-function $A_+(\alpha)$ and the steepness-function $S_+(\alpha)$ are unbounded in the limit $\alpha \to 0$. Equations (18a,c,d) yield

Fig. 1a – The Stationary-Phase Value $t_-(\alpha)$, the Phase-Function $\Theta_-(\alpha)$, the Amplitude-Function $A_-(\alpha)$ and the Steepness-Function $S_-(\alpha)$ Corresponding to the Transverse Waves in the Kelvin Wake
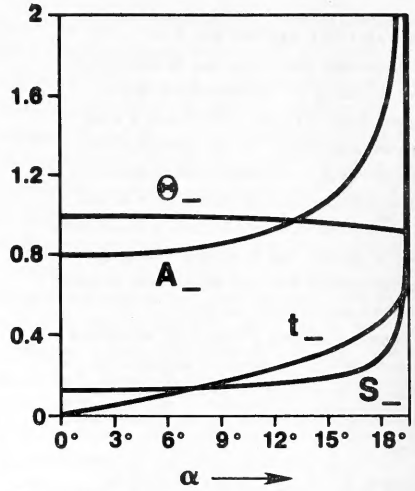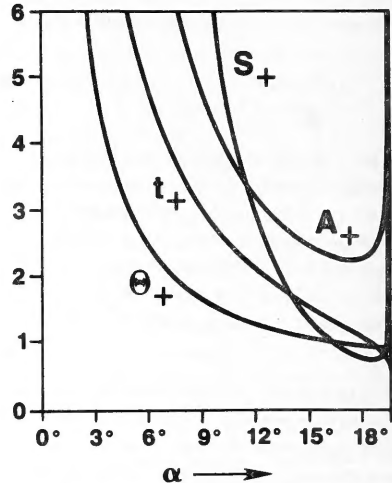

Fig. 1b – The Stationary-Phase Value $t_+(\alpha)$, the Phase-Function $\Theta_+(\alpha)$, the Amplitude-Function $A_+(\alpha)$ and the Steepness-Function $S_+(\alpha)$ Corresponding to the Divergent Waves in the Kelvin Wake

$$A_+ \sim (2/\pi)^{1/2}t_+^{3/2} \text{ and } \qquad (21a)$$

$$S_+ \sim t_+^{7/2}/2^{1/2}\pi^{3/2} \text{ as } t_+ \to \infty. \qquad (21b)$$

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Equations (21a) and (6) show that the amplitude of the divergent waves, given by $A_+|K(t_+)|/(-\xi)^{1/2}$, vanishes at the track of the ship if we have

$$t^{3/2}K(t) \to 0 \text{ as } t \to \infty. \tag{22}$$

Furthermore, it is shown in Barnell and Noblesse (1986) that the asymptotic approximation (6) is uniformly valid at the track of the ship if condition (22) is verified. It may be seen from equations (21b) and (15) that the steepness of the divergent waves, given by $S_+|K(t_+)|/(-\xi)^{1/2}$, is unbounded at the track of the ship if we have

$$t^{7/2}|K(t)| \to \infty \text{ as } t \to \infty. \tag{23}$$

Both conditions (22) and (23) can be satisfied simultaneously if we have

$$|K(t)| \sim 1/t^\mu \text{ as } t \to \infty \text{ with } 3/2 < \mu < 7/2. \tag{24}$$

In summary, the asymptotic approximation (6) expresses the far-field wave pattern of a ship at a point (ξ,α), with $\xi \ll -1$ and $0 \leqslant \alpha < 19°28'$, as the sum of a transverse wave and a divergent wave. The phase $\xi\Theta_\pm(\alpha)\pm\pi/4$ of these two waves are defined explicitly in terms of ξ and α; and their amplitudes $A_\pm(\alpha)K(t_\pm)/(-\xi)^{1/2}$ are given by the product of the functions $A_\pm(\alpha)/(-\xi)^{1/2}$, which are also defined explicitly in terms of ξ and α, and the far-field wave-amplitude function $K[t_\pm(\alpha)]$, which depends on the speed (Froude number) and the shape of the ship. The far-field wave-amplitude function is now considered.

3. BASIC EXPRESSIONS FOR THE FAR-FIELD WAVE-AMPLITUDE FUNCTION

The far-field wave-amplitude function $K(t)$ may be conveniently defined in terms of the nondimensional near-field coordinates $(x,y,z) = (X,Y,Z)/L$, where L is the length of the ship. In the Neumann-Kelvin theory, the function $K(t)$ is given by the sum of an integral around the mean waterline of the ship and an integral over the mean wetted-hull surface. Specifically, for a ship with port and starboard symmetry, as is considered here, the function $K(t)$ may be expressed in the form

$$K(t) = K_+(t) + K_-(t), \tag{25}$$

where the functions $K_\pm$ are given by

$$F^2K_\pm(t) = \int_c E_\pm(n_x^2 + t_x\phi_t - n_zt_y\phi_d + iv^2p\phi)t_y \, dl$$
$$+ v^2\int_h \exp(v^2p^2z)E_\pm(n_x + v^2p^2\phi n_\pm) \, da, \tag{26}$$

as is shown in Noblesse (1983). In this equation, F is the Froude number defined as

$$F = U/(gL)^{1/2}, \tag{27}$$

v is its inverse, that is

$$v = 1/F, \tag{28}$$

p is defined as

$$p = (1+t^2)^{1/2}, \tag{29}$$

and $E_\pm$ represents the exponential function

$$E_\pm = \exp[-iv^2p(x\pm ty). \tag{30}$$

Furthermore, c and h represent the positive halves of the mean waterline and of the mean wetted-hull surface, respectively. The unit vector tangent to c and pointing towards the bow is denoted by $\vec{t}(t_x,t_y,0)$, and $\vec{n}(n_x,n_y,n_z)$ is the unit vector normal to h and pointing into the water, as is indicated in Figure 2. The term $n_\pm$ is defined as

$$n_\pm = -n_z + i(n_x\pm tn_y)/p. \tag{31}$$
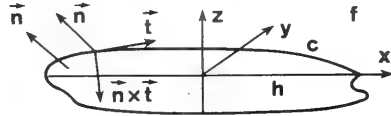
Figure 2 – Definition Sketch for a Single-Hull Ship with Port and Starboard Symmetry

Also, dl and da represent the differential elements of arc length of c and of area of h, respectively. Finally, $\phi = \phi(\vec{x})$ represents the nondimensional disturbance potential $\phi = \Phi/UL$ at the integration point $\vec{x}$ on c or h, ϕ_t represents the derivative of ϕ in the direction of the tangent vector $\vec{t}$ to c, and ϕ_d is the derivative of ϕ in the direction of the vector $\vec{n}\times\vec{t}$, which is tangent to h and pointing downwards as is shown in Figure 2.

Equations (25) and (26) express the far-field wave-amplitude function $K(t)$ in terms of the value of the Froude number and the form of the mean wetted-hull surface, as was noted previously. More precisely, equation (26) expresses the function $K_\pm(t)$ as the sum of a line integral around the mean waterline and a surface integral over the mean wetted-hull surface of the ship. Furthermore, these integrals involve the disturbance velocity potential ϕ in their integrands. The relationship between the far-field wave pattern of a ship and its form

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and speed thus is a complex one, which offers little physical insight. However, analytical approximations for the waterline and the hull integrals in equation (26) can be obtained in the limiting case when the value of the Froude number is sufficiently small. This low-Froude-number asymptotic approximation for the function $K(t)$ is now obtained.

4. LOW-FROUDE-NUMBER LAPLACE APPROXIMATION TO THE FAR-FIELD WAVE-AMPLITUDE FUNCTION IN TERMS OF A WATERLINE INTEGRAL

For large values of $v = 1/F$, or more generally of vp, the exponential function $\exp(v^2p^2z)$ in the hull-surface integral in equation (26) vanishes rapidly for negative values of z. Therefore, only the upper part of the mean-wetted hull surface h yields a significant contribution in the low-Froude-number limit. More precisely, the hull-surface integral can be approximated by a line integral around the mean waterline c. Furthermore, this integral can be combined with the waterline integral in equation (26). The analysis is presented in detail in Noblesse (1986a) and is briefly summarized below.

The mean waterline is represented by the parametric equations

$$x = x_0(\lambda) \text{ and } y = y_0(\lambda), \tag{32a,b}$$

where the parameter λ varies between its bow and stern values, that is

$$\lambda_B \leq \lambda \leq \lambda_S. \tag{32c}$$

In the vicinity of the mean free surface, the hull surface is represented by the parametric equations

$$x = x_0(\lambda) + z\, x_1(\lambda) + z^2 x_2(\lambda) + \dots, \tag{33a}$$

$$y = y_0(\lambda) + z\, y_1(\lambda) + z^2 y_2(\lambda) + \dots, \tag{33b}$$

where $\lambda_B \leq \lambda \leq \lambda_S$ and $z \leq 0$. \hfill (33c)

The velocity potential $\phi(\lambda,z)$ on the hull surface in the vicinity of the plane $z = 0$ likewise is expressed in the form

$$\phi = \phi_0(\lambda) + z\, \phi_1(\lambda) + z^2 \phi_2(\lambda) + \dots. \tag{34}$$

Differentiation of the functions $x_n(\lambda)$, $y_n(\lambda)$, $\phi_n(\lambda)$ with respect to the parameter λ is denoted by the superscript '; thus, we have $x_0' = dx_0(\lambda)/d\lambda$.

By using the foregoing parametric representations for the hull surface and the velocity potential, applying the Laplace method for approximating the hull-surface integral in equation (26), and combining the resulting waterline integral with that already present in equation (26), we may obtain ____ after lengthy algebraic transformations ____ the following low-Froude-number approximation for the function $K(t)$:

$$K(t) \sim v^2q^2 \int_{\lambda_B}^{\lambda_S} (E_0{}^+ a_+ + E_0{}^- a_-)\, d\lambda \text{ as } F \to 0, \tag{35}$$

where we have

$$q = 1/p = 1/(1+t^2)^{1/2}, \tag{36}$$

$E_0{}^\pm$ is the exponential function

$$E_0{}^\pm = \exp[-iv^2p(x_0 \pm ty_0)], \tag{37}$$

and $a_\pm$ is the amplitude function

$$a_\pm = u_\pm a_1{}^\pm + F^2q^2(u_\pm)^2 a_2{}^\pm + O(F^4), \tag{38}$$

where $u_\pm$ is defined as

$$u_\pm = 1/[1 - iq(x_1 \pm ty_1)], \tag{39}$$

and the functions $a_1{}^\pm$ and $a_2{}^\pm$ are now defined.

The first-order amplitude function $a_1{}^\pm$ is given by

$$a_1{}^\pm = y_0' A_\pm/(1+\varepsilon^2) + 2q(x_0' \pm ty_0')(u_\pm)^2 B_\pm \phi_0$$
$$+ C_\pm \phi_0' + uD_\pm \phi_1/(1+\varepsilon^2) + ip(y_1\phi_0' - y_0'\phi_1), \tag{40}$$

where we have

$$\varepsilon = (y_0'x_1 - x_0'y_1)/u, \tag{41}$$

with u defined as

$$u = [(x_0')^2 + (y_0')^2]^{1/2}, \tag{42}$$

and the coefficients $A_\pm$, $B_\pm$, $C_\pm$ and $D_\pm$ are defined as

$$A_\pm = [(1 + py_0'/u)(1 - py_0'/u) + \varepsilon^2]$$
$$+ i(py_0'/u)[y_1(x_0' \pm ty_0')/u + \varepsilon], \tag{43a}$$

$$B_\pm = q(y_2 \mp tx_2) + i(y_1x_2 - x_1y_2), \tag{43b}$$

$$C_\pm = [(\mp t - p^2 x_0'y_0'/u^2)$$
$$+ (py_0'/u)\varepsilon(x_1x_0' + y_1y_0')/u(1+\varepsilon^2)]$$
$$+ i[y_1(x_0' \pm ty_0')/u + \varepsilon]$$
$$[px_0'/u - (py_0'/u)\varepsilon(x_1x_0' + y_1y_0')/u(1+\varepsilon^2)], \tag{43c}$$

$$D_\pm = [(x_0' \pm ty_0')/u][(1+\varepsilon^2)(y_1 \pm iqt)u_\pm + i(py_0'/u)\varepsilon y_1]$$
$$- (py_0'/u)\varepsilon(py_0'/u - i\varepsilon). \tag{43d}$$

The second-order amplitude function $a_2{}^\pm$ is given by

$$a_2{}^\pm = -y_1' + 2u_\pm(\phi_0 m_2{}^\pm + \phi_1 m_1{}^\pm + \phi_2 m_0{}^\pm - iy_0'\gamma_2{}^\pm)$$
$$+ 6i(u_\pm)^2[\phi_0(m_1{}^\pm\gamma_2{}^\pm + m_0{}^\pm\gamma_3{}^\pm) + \phi_1 m_0{}^\pm\gamma_2{}^\pm]$$
$$- 12(u_\pm)^3\phi_0 m_0{}^\pm(\gamma_2{}^\pm)^2, \tag{44}$$

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where we have

$$\gamma_n^\pm = q(x_n \pm ty_n),$$ (45)

$$m_n^\pm = \mu_n + iq(y_n' \mp tx_n'),$$ (46)

with

$$\mu_0 = \varepsilon u,$$ (47a)

$$\mu_1 = x_1 y_1' - y_1 x_1' + 2(x_2 y_0' - y_2 x_0'),$$ (47b)

$$\mu_2 = x_1 y_2' - y_1 x_2' + 2(x_2 y_1' - y_2 x_1')$$
$$+ 3(x_3 y_0' - y_3 x_0').$$ (47c)

In the particular case when the phase of the exponential function $E_0^\pm$ is stationary, that is at a point $(x_0, y_0, 0)$ where we have $x_0' \pm ty_0' = 0$, the first-order amplitude function $a_1^\pm$ takes the form

$$a_1^\pm = \pm \varepsilon[iy_0' + (py_1 \mp it)\phi_0' \mp u\phi_1]/(1 \pm i\varepsilon)$$
$$+ ip(y_1'\phi_0' - y_0'\phi_1) \quad \text{if } x_0' \pm ty_0' = 0.$$ (48)

Furthermore, if the hull surface intersects the plane $z = 0$ orthogonally at a point of stationary phase we have

$$a_1^\pm = i F^2 py_0' \partial^2 \phi_0 \partial x^2$$
$$\text{if } x_0' \pm ty_0' = 0 \text{ and } n_z = 0,$$ (49)

and the amplitude function $a_\pm$ then is or order F^2.

The low-Froude-number asymptotic approximation to the far-field wave-amplitude function $K(t)$ given by the waterline integral (35) is considerably simpler than the exact expression (26), which involves both a waterline integral and a hull-surface integral; and the approximate expression (35) is well suited for efficient numerical evaluation. However, expression (35) can be simplified by applying the method of stationary phase, which takes advantage of the rapid oscillations of the exponential function $E_0^\pm$ given by equation (37) in the low-Froude-number limit $v \to \infty$, or more generally in the limit $vp \to \infty$. This stationary-phase approximation is now obtained.

5. LOW-FROUDE-NUMBER STATIONARY-PHASE APPROXIMATION TO THE FAR-FIELD WAVE-AMPLITUDE FUNCTION

The method of stationary phase indicates that the major contributions to the integral (35) in the limit when the exponential functions $E_0^\pm$ are rapidly oscillating, that is if $v \to \infty$ or more generally $vp \to \infty$, stem from points where the phases of these exponential functions are stationary, that is from points where we have $x_0' \pm ty_0' = 0$, and from the end points λ_B and λ_S of the integration

range, that is from the bow and the stern. This stationary-phase analysis is presented in detail in Noblesse (1986b) and only its results are given here. We have

$$K(t) \sim iq^3(K_B - K_S + \Sigma K_\pm) \text{ as } F \to 0,$$ (50)

where K_B and K_S correspond to the contributions of the bow and the stern, respectively, and $K_\pm$ corresponds to the contribution of a point where the phase of the exponential function $E_0^\pm$ is stationary, that is where we have $y_0'/(-x_0') = \pm 1/t$ or $dy_0/dx_0 = \mp 1/t$; the summation in the asymptotic approximation (50) is extended over all such points of stationary phase on the mean waterline c. The expressions for the contributions of the bow and stern $K_{B,S}$ and of the points of stationary phase $K_\pm$ are given below.

The contribution $K_{B,S}$ of the bow or stern may be expressed in the form

$$K_{B,S} = A_{B,S} \exp(-iv^2 px_{B,S}),$$ (51)

where $x_{B,S}$ is the abscissa of the bow or stern and the amplitude-function $A_{B,S}$ is given by

$$A_{B,S} = 2n_x A_1 + iF^2 q(A_2^+ + A_2^-) + O(F^4);$$ (52)

in this expression, the functions A_1 and $A_2^\pm$ are defined by the equations

$$(n_x + iqn_z)A_1 = n_x n_y[1 - ipn_z(n_x + iqn_z)/(1 - n_z^2 - p^2 n_x^2)]$$
$$+ t_y \phi_t - ipn_y t_y \phi_d (1 - n_z^2 + ipn_x n_z)/(1 - n_z^2 - p^2 n_x^2)$$ (53)

and

$$\theta_\pm' A_2^\pm = (u_\pm a_1^\pm/\theta_\pm')' + iq(u_\pm)^2 a_2^\pm,$$ (54)

where the superscript ' denotes differentiation with respect to the parameter λ, the functions $u_\pm$, $a_1^\pm$ and $a_2^\pm$ are defined by equations (39), (40) and (44), respectively, and the function $\theta_\pm$ is defined as

$$\theta_\pm = x_0 \pm ty_0.$$ (55)

Expression (54) for the second-order amplitude-function $A_2^\pm$ is a complex one. However, equation (53) defines the first-order amplitude function A_1 explicitly in terms of the value of t, $p = (1 + t^2)^{1/2}$ and $q = 1/p$, the geometrical characteristics of the hull at the bow or stern, namely the unit vector $\overline{t}(t_x, t_y, 0)$ tangent to the mean waterline and the unit vector $\overline{n}(n_x, n_y, n_z)$ normal to the hull, and the components ϕ_t and ϕ_d of the velocity vector in the directions of the unit vectors $\overline{t}$ and $\overline{n} \times \overline{t}$ tangent to the hull. Equation (52) shows that the first-order approximation to the amplitude-function $A_{B,S}$,

given by $2n_x A_1$, vanishes if $n_x = 0$, that is if the waterline has a cusp at the bow or stern. Equation (53) shows that the first-order approximation to the function $A_{B,S}$ also vanishes if the bow or stern is round, since we then have $n_y = 0$ and $\phi_t = -\phi_y = 0$ by symmetry. It may thus be seen that the contribution of the bow or stern to the far-field wave-amplitude function $K(t)$ is of order F^2 if $n_x = 0$ or $n_y = 0$ at the bow or stern, respectively; that is, we have

$$K_{B,S} = O(F^2) \quad \text{if } n_x = 0 \text{ or } n_y = 0 \tag{56}$$

at the bow or stern. In the particular case when the hull surface is vertical at the bow or stern we have $n_z = 0$ and equation (53) becomes

$$A_1 = t_x - \phi_t - ipt_x\phi_z/(1 - p^2 t_y^2) \quad \text{if } n_z = 0, \tag{57}$$

which yields

$$A_1 = t_x - \phi_t + O(F^2) \quad \text{if } n_z = 0 \tag{58}$$

as is indicated by the free-surface boundary condition $\phi_z = -F^2\phi_{xx} = O(F^2)$.

The contribution $K_\pm$ of a point of the mean waterline c where the phase of the exponential function $E_0^\pm$ is stationary, that is where we have

$$dy_0/dx_0 = \mp 1/t, \tag{59}$$

may be expressed in the form

$$K_\pm = \pm v(2\pi r)^{1/2} A_\pm$$
$$\exp \left[\mp i v^2(y_0 t_x - x_0 t_y)/t_y^2 + i\varepsilon\pi/4\right], \tag{60}$$

where r is the radius of curvature of c at the point of stationary phase (x_0, y_0), ε is equal to $+1$ or -1 if the center of curvature of c at the point (x_0, y_0) is upstream or downstream from (x_0, y_0), respectively, and the amplitude-function $A_\pm$ is given by

$$A_\pm = (1 - n_z^2)^{1/2}(n_z t_y + \phi_d) - F^2 q A_2^\pm + O(F^4); \tag{61}$$

in this expression, the second-order amplitude-function $A_2^\pm$ is defined by the equation

$$\pm 2[(x_0')^2 + (y_0')^2]^{1/2} A_2^\pm = [(u_\pm a_1^\pm)'/\theta_\pm'']'$$
$$+ u_\pm a_1^\pm [5(\theta_\pm''')^2/3\theta_\pm'' - \theta_\pm^{(4)}]/4(\theta_\pm'')^2$$
$$+ 2iq(u_\pm)^2 a_2^\pm, \tag{62}$$

where the superscript $'$ denotes differentiation with respect to the parameter λ and the functions $u_\pm$, $a_1^\pm$, $a_2^\pm$ and $\theta_\pm$ are given by equations (39), (40), (44) and (55), respectively.

The expression for the second-order amplitude-function $A_2^\pm$ is a complex one. However, equation (60) and the first-order approximation to the amplitude function, namely

$$A_\pm = (1 - n_z^2)^{1/2}(n_z t_y + \phi_d) + O(F^2), \tag{63}$$

provide a simple explicit expression for the stationary-phase contribution $K_\pm$ in terms of the geometric characteristics of the hull and the downward tangential derivative ϕ_d of the potential at the point of stationary phase. In the particular case when the hull surface is vertical at the point of stationary phase we have $n_z = 0$ and equation (63) becomes

$$A_\pm = -\phi_z + O(F^2) \quad \text{if } n_z = 0, \tag{64}$$

which yields

$$A_\pm = O(F^2) \quad \text{if } n_z = 0. \tag{65}$$

Equation (60) then shows that the stationary-phase contribution at a point (x_0, y_0) of c where the hull is vertical is of order F, that is we have

$$K_\pm = O(F) \quad \text{if } n_z = 0. \tag{66}$$

On the other hand, equations (60) and (63) show that we have

$$K_\pm = O(1/F) \quad \text{if } n_z \neq 0. \tag{67}$$

The stationary-phase contribution at a point where the hull has flare thus is dominant in the zero-Froude-number limit.

The summation in equation (50) is extended to all the points of the mean waterline c where the phase of the exponential function E_0^+ or the function E_0^- is stationary, that is the points where the slope dy_0/dx_0 of c is equal to $-1/t$ or $+1/t$, respectively. The number of stationary points, and their position along the waterline, depend on the value of t and on the shape of c. For instance, for the simple case of a hull with waterline consisting of a sharp-ended parabolic bow region $1/4 \leqslant x \leqslant 1/2$ defined by the equation $y = 4bx(1 - 2x)$, where b denotes the ship's beam/length ratio, a straight parallel midbody region $-1/4 \leqslant x \leqslant 1/4$, and a round-ended elliptic stern region $-1/2 \leqslant x \leqslant -1/4$ defined by the equation $y = b[-2x(1 + 2x)]^{1/2}$, there is one point of stationary phase in the stern region given by $x = -[1 + 1/(1 + 4b^2 t^2)^{1/2}]/4$, so that we have $-1/2 \leqslant x \leqslant -1/4$ for $0 \leqslant t \leqslant \infty$ with $x \to -1/2$ as $t \to 0$ and $x \to -1/4$ as $t \to \infty$, and one point of stationary phase in the

bow region given by $x = (1 + 1/4bt)/4$ for $1/4b \leqslant t \leqslant \infty$, so that we have $1/2 \geqslant x \geqslant 1/4$ for $1/4b \leqslant t \leqslant \infty$ with $x \rightarrow 1/2$ as $t \rightarrow 1/4b$ and $x \rightarrow 1/4$ as $t \rightarrow \infty$. We thus have one point of stationary phase in the stern region for $0 \leqslant t < 1/4b$ and two points of stationary phase, one in the stern region and one in the bow region, for $1/4b \leqslant t \leqslant \infty$. The two points of stationary phase approach the shoulders $x = \pm 1/4$, where $dy/dx = 0$, as $t \rightarrow \infty$.

The asymptotic approximation (50) and equations (51)-(58) and (59)-(67) defining the contributions of the bow and stern and of the stationary-phase point(s) on the waterline, respectively, show that the low-Froude-number behavior of the far-field wave-amplitude function is strongly influenced by the shape of the hull in the vicinity of the waterline. More precisely, for a value of t for which there is one (or more) point of stationary phase on the mean waterline where the hull has flare, the contribution of this stationary-phase point dominates the contribution of the bow and stern and is of order $1/F$, that is we have $K(t) = O(1/F)$. On the other hand, for a value of t for which either there corresponds no point of stationary phase or the hull has no flare at the point(s) of stationary phase, the dominant contribution stems from the bow and stern, and it is of order 1, that is we then have $K(t) = O(1)$. However, if $n_x = 0$ or $n_y = 0$ at both the bow and the stern, that is if the bow and the stern are either cusped or round, their contribution is $O(F^2)$ and the contribution of the stationary-phase point(s), which is $O(F)$ if there is no flare (as is assumed here), is dominant; so that we then have $K(t) = O(F)$.

For a ship form that is everywhere wall sided, the contribution of the bow and stern is dominant for all values of t, and we have $K(t) = O(1)$ for $0 \leqslant t \leqslant \infty$. On the other hand, for a hull form that has flare over a portion of the waterline and is wall sided elsewhere, the contribution of the bow and stern is dominant, and $O(1)$, only for those values of t for which the corresponding points of stationary phase fall outside the range of flare; for the range of values of t for which the corresponding points of stationary phase are within the range of flare, the contribution of these stationary-phase points is dominant, of order $1/F$. In this instance, the function $K(t)$ is $O(1/F)$ for a range of values of t (corresponding to the region of flare) and $O(1)$ for other values to t. If the region of flare is of small extent, and the slope dy/dx of the waterline does not vary widely within that

region, the range of values of t for which $K(t) = O(1/F)$ is also small. The far-field wave-amplitude function $K(t)$ can then exhibit a sharp peak for some value of t in the low-Froude-number limit. In fact, several isolated peaks of the function $K(t)$ can exist if the hull form has several distinct regions of flare within which the slope of the waterline varies gradually.

It should be noted that the result $K(t) = O(1/F)$ for values of t for which the hull has flare at the corresponding points of stationary phase does not imply that the corresponding free-surface elevation becomes unbounded in the zero-Froude-number limit $F = 0$. Indeed, the asymptotic approximation (6), where we have $(\xi, \zeta) = (X, Z)g/U^2$, then yields $Zg/U^2 = O[1/F(-\xi)^{1/2}] = O[1/(-X/L)^{1/2}]$ as $F \rightarrow 0$ and $-X/L \rightarrow \infty$. The free-surface elevation Z thus is of order $(U^2/g)/(-X/L)^{1/2}$ as $F \rightarrow 0$ and $-X/L \rightarrow \infty$, and equation (15) shows that the corresponding wave steepness s is $O[1/(-X/L)^{1/2}]$. For values of t for which $K(t) = O(1)$ it is seen that Zg/U^2 and s are $O[F/(-X/L)^{1/2}]$ as $F \rightarrow 0$ and $-X/L \rightarrow \infty$.

It should also be noted that the asymptotic approximation (51)-(55) for the contribution of the bow and stern is not uniformly valid for the values of t for which the bow or the stern is a point of stationary phase, that is for which the waterline slope dy_0/dx_0 at the bow or the stern is equal to $-1/t$ or $1/t$, respectively. Indeed, it may be shown that we have $1 - n_z^2 - p^2 n_x^2 = 0$ at a point of stationary phase, so that the first-order approximation to the amplitude functions $A_{B,S}$ given by equation (53) becomes unbounded. The asymptotic approximation (60)-(62) for the contribution of a point of stationary phase likewise is not uniformly valid at a stationary-phase point where th waterline has an inflexion point. Indeed, the radius of curvature r at such an inflexion point is infinite and equation (60) yields an unbounded contribution $K_\pm$. Asymptotic approximations valid for these special cases may easily be obtained and are given in Noblesse (1986b). It will only be noted here that the far-field wave-amplitude function $K(t)$ for a hull form having flare at a point where the waterline has an inflexion may be expected to exhibit a particularly pronounced peak at the value of t corresponding to the inflexion point since we have $K(t) = O(1/F^{4/3})$ as $F \rightarrow 0$ for this particular value of t.

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6. CONCLUSION: HULL FORM AND KELVIN-WAKE FEATURES

The classical far-field asymptotic approximation to the Kelvin wake, obtained in section 2 by applying the method of stationary phase, and the low-Froude-number asymptotic approximation to the far-field wave-amplitude function, obtained in sections 4 and 5 by successively using the Laplace method and the method of stationary phase, provide a simple analytical relationship between the hull shape, on one hand, and the waves it generates, on the other hand. This explicit relationship between the wavemaker and its waves is summarized below.

The far-field asymptotic approximation to the Kelvin wake (6) shows that at any point (ξ, α), with $\xi = Xg/U^2 \ll -1$ and $0 \leqslant \alpha < \tan^{-1}(2^{-3/2}) \simeq 19°28'$, the wave field consists in two plane progressive waves _____ a transverse wave and a divergent wave _____ with wavelengths λ_- and λ_+ propagating at angles β_- and β_+ from the track of the ship, respectively. The wavelengths $\lambda_\pm$ and the propagation angles $\beta_\pm$ depend on the angle from the ship track α alone, that is $\lambda_\pm$ and $\beta_\pm$ are independent of the hull shape and size, as is well known. Specifically, the functions $\lambda_\pm(\alpha)$ and $\beta_\pm(\alpha)$ are defined by equations (10), (11) and (12). At a given downstream distance ξ, the amplitudes of these waves, on the other hand, are given by the product of the functions $A_\pm(\alpha)$, defined by equation (9), and the far-field wave-amplitude function K(t) evaluated at the stationary values $t_\pm(\alpha)$ given by equation (7). The function K(t) depends on the hull shape and the Froude .number in a fairly complicated manner via an integral over the mean wetted-hull surface and an integral around the mean waterline, as is indicated by equation (26). A low-Froude-number asymptotic approximation to these integrals is obtained in sections 4 and 5.

The analytical approximation (50) shows that for a given value of t corresponding to a given value of α, as is specified by equation (7), the main contributions to the function K(t) stem from several particular points on the mean waterline. These are the bow and the stern, on one hand, and (usually but not always) one (or several) point(s) of stationary phase. Indeed, the number of these points of stationary phase and their position on the waterline, defined by the condition

$$|dy_0/dx_0| = 1/t, \tag{68}$$

depend on the shape of the waterline and the value of t.

Let φ denote the angle between the tangent to the mean waterline and the track of the ship, that is we have

$$\tan\varphi = |dy_0/dx_0| \quad \text{and} \quad 0 \leqslant \varphi \leqslant \pi/2. \tag{69}$$

Equations (68), (69), (7) and (10) then show that the stationary values $t_+(\alpha)$ and $t_-(\alpha)$ associated with a given value of α are defined by the equation

$$\varphi_\pm(\alpha) = \tan^{-1}\{4\sigma/[1 \pm (1 - 8\sigma^2)^{1/2}]\}, \tag{70}$$

where $\sigma = \tan\alpha$. For a given waterline shape, equation (70) thus defines the number of stationary points and their position on the waterline corresponding to any given angle α inside the Kelvin wake. In particular, equation (70) yields

$$0 \leqslant \varphi_+ \leqslant \tan^{-1}(2^{1/2}) \simeq 54°44' \leqslant \varphi_- \leqslant 90° \tag{71}$$

and $\varphi_+ \sim 2\alpha$, $\varphi_- \sim \pi/2 - \alpha$ as $\alpha \to 0$. (72a,b)

Points of the waterline with slope between 0 and 54°44' thus contribute mostly to the system of divergent waves while waterline slopes between 54°44' and 90° mostly contribute to the transverse waves.

Equations (10), (11), (12) and (70) define the wavelengths $\lambda_\pm$, the wave-propagation angles $\beta_\pm$ and the waterline-tangent angles $\varphi_\pm$ corresponding to a given angle from the ship track α. The functions $\lambda_\pm(\alpha)$, $\beta_\pm(\alpha)$ and $\varphi_\pm(\alpha)$ are depicted in Figure 3a, where the subscripts T and D are used, instead of $-$ and $+$, to refer to the transverse and divergent waves, respectively. The foregoing relationships between α and λ, β, φ may be used for determining the angle from the ship track α, the wavelength λ and the wave-propagation angle β corresponding to a given waterline-tangent angle φ. Specifically, we may obtain the remarkably simple relations

$$\alpha = \tan^{-1}[\tan\varphi/(2 + \tan^2\varphi)], \tag{73}$$

$$\lambda/2\pi = \sin^2\varphi \quad \text{and} \quad \beta = \pi/2 - \varphi. \tag{74a,b}$$

The functions $\alpha(\varphi)$, $\lambda(\varphi)$ and $\beta(\varphi)$ are depicted in Figure 3b. Alternatively, the foregoing relationships among φ, α, λ and β can be represented in the form of Figures 3c and 3d, which depict the functions $\varphi(\beta)$, $\alpha(\beta)$, $\lambda(\beta)$ and $\varphi(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$, respectively. These equivalent graphical representations show that we have

$$0 \leqslant \varphi_D \leqslant \tan^{-1}(2^{1/2}) \simeq 54°44' \leqslant \varphi_T \leqslant 90°, \tag{75a}$$

$$0 \leqslant \lambda_D/2\pi \leqslant 2/3 \leqslant \lambda_T/2\pi \leqslant 1, \tag{75b}$$

$$90° \geqslant \beta_D \geqslant \tan^{-1}(2^{-1/2}) \simeq 35°16' \geqslant \beta_T \geqslant 0, \tag{75c}$$
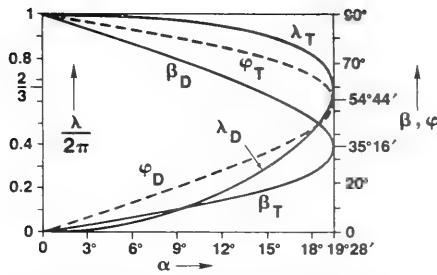
Fig. 3a – The Waterline-Tangent Angle φ, the Wave-Propagation Angle β and the Wavelength λ as Functions of the Angle From the Ship Track α; the Subscripts T and D Refer to the Transverse and Divergent Waves, Respectively.

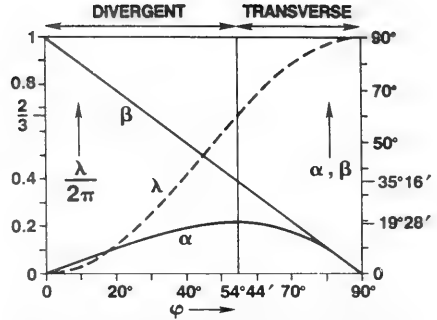Fig. 3b – The Angle From the Ship Track α, the Wave-Propagation Angle β and the Wavelength λ as Functions of the Waterline-Tangent Angle φ.
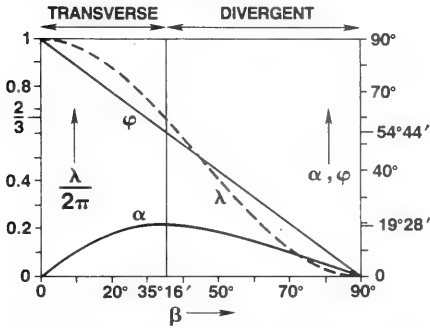
Fig. 3c – The Waterline-Tangent Angle φ, the Angle From the Ship Track α and the Wavelength λ as Functions of the Wave-Propagation Angle β.
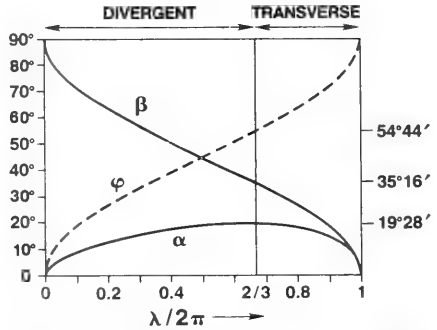
Fig. 3d – The Waterline-Tangent Angle φ, the Angle From the Ship Track α and the Wave-Propagation Angle β as functions of the Wavelength λ.

where the subscripts D and T refer to the divergent and transverse waves, respectively.

The first two terms in the low-Froude-number asymptotic expansions for the contributions $K_{B,S}$ of the bow and stern and the contributions $K_{\pm}$ of the points of stationary phase in equation (50) are given by equations (51)-(55) and (59)-(62), respectively. The second-order terms in these asymptotic expansions are defined by complex expressions. However, the first-order terms provide simple approximations defined explicitly in terms of the geometrical characteristics of the hull and the velocity components in the tangential directions $\vec{t}$ and $\vec{n} \times \vec{t}$ to the hull. In particular, the low-Froude-number asymptotic expansions given in section 5 show that the contributions K_B and K_S of the bow and stern are $O(1)$ except if the bow or stern is cusped or round, in which

case we have $K_{B,S} = O(F^2)$. The contribution of a given point of stationary phase is $O(1/F)$, and thus is dominant, if the hull has flare at this point; otherwise, that is if the hull is wall sided at the point of stationary phase, its contribution is $O(F)$.

Thus, for a ship form that is everywhere wall sided, the contributions of the bow and stern are dominant (assuming that they are not both either round or cusped) for all values of the angle α from the ship track, that is everywhere in the far-field Kelvin wake. On the other hand, for a hull that has flare over a portion of the waterline and is wall sided elsewhere, the contributions of the bow and stern are dominant, and $O(1)$, only for those angles α in the Kelvin wake for which the corresponding points of stationary phase on the waterline fall outside the region of flare; for the

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range of values of α for which the corresponding points of stationary phase are within the range of flare, the contribution of these stationary-phase points is dominant, of order $1/F$. In this instance, the amplitude of the waves in the far-field Kelvin wake is of order $(U^2/g)/(-X/L)^{1/2}$, as $F \to 0$ and $X/L \to -\infty$, for the range of values of α corresponding to the region of flare and of order $F(U^2/g)/(-X/L)^{1/2}$ for values of α outside this range. If the region of flare is of small extent, and the waterline-tangent angle φ does not vary widely within that region, the corresponding range of values of the angle α where the wave amplitude is an order of magnitude larger than elsewhere is also small, and thus appears as a peak for sufficiently small values of the Froude number. This peak is particularly pronounced for a hull with a small region of flare in the vicinity of an inflexion point of the waterline.

The low-Froude-number asymptotic analysis of the Neumann-Kelvin theory presented in this study thus shows that the characteristics of the far-field Kelvin wake strongly depend on the shape of the ship hull, notably the presence of flare and the shape of the waterline at the bow and stern. This analysis also predicts that the nondimensional wave-resistance coefficient $R/\varrho U^2 L^2$, where U and L are the speed and the length of the ship and ϱ is the density of water, which is given by the Havelock integral

$$\pi R/\varrho U^2 L^2 \;=\; F^4 \int_0^\infty |K(t)|^2 (1+t^2)^{-1/2}\, dt, \tag{76}$$

is $O(F^2)$ for a ship form with a region of flare, $O(F^4)$ for a ship form that is wall sided everywhere and has either a bow or a stern (or both) that is neither cusped nor round, and $O(F^6)$ for a wall-sided ship form with both bow and stern that are either cusped or round.

The low-Froude-number asymptotic approximation (50) also shows that we have

$$K(t) \;=\; O(1/t^3) \text{ as } t \to \infty. \tag{77}$$

This result is actually valid for any value of the Froude number; indeed, the asymptotic approximation (50) is valid not only in the low-Froude-number limit $F \to 0$ but more generally in the limit $vp = (1+t^2)^{1/2}/F \to \infty$, that is as $F \to 0$ or/and as $t \to \infty$, as may be seen from the exponential functions $\exp(v^2 p^2 z)$ and $E_\pm = \exp[-iv^2 p(x \pm ty)]$ in equation (26).

Equations (15), (18a), (21b) and (77) then yield

$$(-\xi)^{1/2} s_+(\xi,\alpha) \sim t_+^{1/2} \sim 1/\alpha^{1/2} \text{ as } \alpha \to 0 \tag{78}$$

for the steepness of the short divergent waves in the vicinity of the track of the ship. By using equations (4) and (5), which yield $\alpha \sim \eta/(-\xi)$ as $\alpha \to 0$, in equation (78) we may obtain

$$s_+(\xi,\eta) \sim 1/\eta^{1/2} \text{ as } \eta \to 0. \tag{79}$$

Equation (79) thus shows that the lines along which the steepness of the short divergent waves in the far-field Kelvin wake takes given large values, say $s_+ = 1/7$ and $1/15$, are parallel to the ship track, as was found in Figure 21 of Barnell and Noblesse (1986) by using the Michell thin-ship approximation for a simple ship form. The Neumann-Kelvin theory therefore predicts that the far-field Kelvin wake contains three distinct regions: (i) a narrow constant-width inner region bordering the track of the ship where no divergent gravity waves can exist, (ii) an outer region where the usual transverse and divergent waves are present, and (iii) an intermediate region at the boundary between the inner and outer regions where short steep divergent waves can be found. In reality, surface tension must evidently be taken into account in the vicinity of the track of the ship.

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